,

Weak Forms of ω−Opensets

Mohammed Al-Hawmi*, Amin Saif**, Yahya Awbel

*Department of Mathematics,Faculty of Education, University of Saba Region, Mareb, Yemen

**Department of Mathematics, Faculty of Sciences, Taiz University, Taiz, Yemen

ABSTRACT

The principle purpose of this paper is to introduce and study some new classes of sets in topological spaces which are finer than the classes of open sets and $\omega-$open sets. The continuity via these classes will be introduced and studied.

AMS classification: Primary 54A05, 54A10, 54C10.

Keywords

Open set; Generalized Open set; Decomposition of continuity.

1. INTRODUCTION

In general topology, many authors introduced and studied some classes of weak or strong forms of open sets in topological spaces. In 1970 Levine, [6], introduced the notion of a generalized open sets which is a weak form of open sets. In 1982 Hdeib [2] introduced the notion of a $\omega-$open sets as a weak form of open sets. In 1983 the authors [5] introduced the weak form for an open set which is called a $\beta-$open set. In 2005 Al-Zoubi [4] introduced the generalization property of $\omega-$open sets to get a weak form of $\omega-$open sets. In 2009 Noiri and Noorani [7] introduced the notion of $\beta\omega-$open set which is a weak form for a $\omega-$open sets and a $\beta-$open sets.

This paper is organized as follows. Section 2 is devoted to some preliminaries. In Section 3 we introduce the concept of generalized $\beta\omega-$open sets by utilizing the $\beta\omega-$closure operator. Furthermore, the relationship with the other known sets will be studied. In Section 4 we introduce the notions of $\beta\omega-$continuous, generalized $\beta\omega-$continuous, Slightly and Contra $\beta\omega-$Continuous functions.

2. PRELIMINARIES

For a topological space (X,τ) and $A\subseteq X$, throughout this paper, we mean $Cl(A)$ and $Int(A)$ the closure set and the interior set of A, respectively.

THEOREM 2.1. [1] For a topological space (X,τ) and $A,B\subseteq X$, if B is an open set in X then $Cl(A)capB\subseteq Cl(A\cap B)$.

THEOREM 2.2. [1] For a topological space (X,τ),

(1) $Cl(X-A)=X-Int(A)$ for all $A\subseteq X$.
(2) $Int(X-A)=X-Cl(A)$ for all $A\subseteq X$.

DEFINITION 2.3. [6] A subset A of a topological space (X,τ) is called *generalized closed* (simply $g-$closed) *set*, if $Cl(A)\subseteq U$ whenever $A\subseteq U$ and U is open subset of (X,τ). The complement of $g-$closed set is called *generalized open* (simply $g-$open) *set*.

THEOREM 2.4. [6] Every closed set is a $g-$closed set.

DEFINITION 2.5. A topological space (X,τ) is called:

(1) $T_{1/2}-$*space* [6] if every $g-$closed set is closed set.
(2) T_1-*space* [1] if for each disjoint point $x\neq y\in X$, there are two open sets G and H in X such that $x\in H$, $y\in G$, $x\notin G$ and $y\notin H$.

THEOREM 2.6. [3] A topological space (X,τ) is $T_{1/2}-$*space* if and only if every singleton set is open or closed set.

THEOREM 2.7. [1] A topological space (X,τ) is T_1-space if and only if every singleton set is closed set.

DEFINITION 2.8. [2] A subset A of a space X is called $\omega-$*open set* if for each $x\in A$, there is an open set U_x containing x such that U_x-A is a countable set. The complement of a $\omega-$open set is called a $\omega-$closed set. The set of all $\omega-$closed sets in X denoted by $\omega C(X,\tau)$ and the set of all $\omega-$open sets in X denoted by $\omega O(X,\tau)$.

THEOREM 2.9. [2] Every open set is $\omega-$open set.

THEOREM 2.10. [2] For a topological space (X,τ), the pair $[X,\omega O(X,\tau)]$ forms a topological space.

For a topological space (X,τ) and $A\subseteq X$, the $\omega-$closure set of A is defined as the intersection of all $\omega-$closed subsets of X containing A and is denoted by $Cl_\omega(A)$. The $\omega-$interior set of A is defined as the union of all $\omega-$open subsets of X contained in A and is denoted by $Int_\omega(A)$.

DEFINITION 2.11. [4] A subset A of a space X is called *generalized $\omega-$closed set* (simply $g\omega-$closed) *set* if $Cl_\omega(A)\subseteq U$ whenever $A\subseteq U$ and U is open set. The complement of generalized $\omega-$closed set is called generalized $\omega-$open set (simply $g\omega-$open) *set*.

THEOREM 2.12. [4] Every $g-$closed set is a $g\omega-$closed set.

DEFINITION 2.13. [7] A subset A of a topological space (X,τ) is called $\beta\omega-$open set if $A\subseteq Cl(Int_\omega(Cl(A)))$. The complement of $\beta\omega-$open set is called $\beta\omega-$closed set. The set of all $\beta\omega-$closed sets in X denoted by $\beta\omega C(X,\tau)$ and the set of all $\beta\omega-$open sets in X denoted by $\beta\omega O(X,\tau)$.

THEOREM 2.14. [7] The union of arbitrary of $\beta\omega-$open sets is $\beta\omega-$open set.

THEOREM 2.15. [7] Every $\omega-$open set is $\beta\omega-$open set.

DEFINITION 2.16. A function $f : (X, \tau) \to (Y, \rho)$ of a space (X, τ) into a space (Y, ρ) is called:

(1) *$g-$continuous function* [6] if $f^{-1}(U)$ is a $g-$open set in X for every open set U in Y.
(2) *$\omega-$continuous function* [2] if for each $x \in X$ and for an open set G in Y containing $f(x)$, there is a $\omega-$open set U in X containing x such that $f(U) \subseteq G$.
(3) *$g\omega-$continuous function* [4] if $f^{-1}(U)$ is a $g\omega-$open set in X for every open set U in Y.

THEOREM 2.17. [2] A function $f : (X, \tau) \to (Y, \rho)$ is a $\omega-$continuous function if and only if $f^{-1}(U)$ is a $\omega-$open set in X for every open set U in Y.

It is clear that Every continuous function is $g-$continuous function [6], every continuous function is $\omega-$continuous function [2], every $\omega-$continuous function is $g\omega-$continuous function [4] and every $g-$continuous function is $g\omega-$continuous function [4].

3. GENERALIZED $\beta\omega-$OPEN SETS

For a topological space (X, τ) and $A \subseteq X$, the $\beta\omega-$closure set of A is defined as the intersection of all $\beta\omega-$closed subsets of X containing A and is denoted by $Cl_{\beta\omega}(A)$. The $\beta\omega-$interior set of A is defined as the union of all $\beta\omega-$open subsets of X contained in A and is denoted by $Int_{\beta\omega}(A)$. From Theorem (2.14), $Cl_{\beta\omega}(A)$ is a $\beta\omega-$closed subsets of X and $Int_{\beta\omega}(A)$ is $\beta\omega-$open subsets of X.

DEFINITION 3.1. A subset A of a topological space (X, τ) is called *generalized $\beta\omega-$closed* (simply $G_{\beta\omega}-$closed) *set*, if $Cl_{\beta\omega}(A) \subseteq U$ whenever $A \subseteq U$ and U is open subset of (X, τ). The complement of $G_{\beta\omega}-$closed set is called *generalized $\beta\omega-$open* (simply $G_{\beta\omega}-$open) *set*.

For a topological space (X, τ), the set of all $G_{\beta\omega}-$closed sets in X denoted by $G_{\beta\omega}C(X, \tau)$ and the set of all $G_{\beta\omega}-$open sets in X denoted by $G_{\beta\omega}O(X, \tau)$.

EXAMPLE 3.2. For any topological space (X, τ), if X is a countable then it's clear that every subset of X is i a both $G_{\beta\omega}-$closed and $G_{\beta\omega}-$open set. That is,

$$G_{\beta\omega}O(X, \tau) = G_{\beta\omega}C(X, \tau) = P(X),$$

where $P(X)$ is the power of X.

EXAMPLE 3.3. Let (R, τ_u) be the real usual topological space on the set of real numbers R. The rational set Q is a $G_{\beta\omega}-$closed set, since the irrational set IR is a $\beta\omega-$open set, that is, $Cl_{\beta\omega}(Q) = Q$.

THEOREM 3.4. Any a countable subset of a topological space (X, τ) is a $G_{\beta\omega}-$closed set in X.

THEOREM 3.5. Every $\beta\omega-$open set is $G_{\beta\omega}-$open set.

COROLLARY 3.6. Every $\beta\omega-$closed set is $G_{\beta\omega}-$closed set.

The converse of the last theorem is no need to be true.

EXAMPLE 3.7. In topological space (R, τ), the set $R-\{2\}$ is $G_{\beta\omega}-$closed set but it is not $\beta\omega-$closed set, where $\tau = \{\emptyset, R, R-\{2, 3\}\}$.

THEOREM 3.8. Let (X, τ) be a topological space. If (X, τ) is a $T_{1/2}-$space then every $G_{\beta\omega}-$closed set in X is $\beta\omega-$closed set in X.

PROOF. Let A be a $G_{\beta\omega}-$closed set in X. Suppose that A is not $\beta\omega-$closed set. Then there is at least $x \in Cl_{\beta\omega}(A)$ such that $x \notin A$. Since (X, τ) is a $T_{1/2}-$space then by Theorem (2.6), $\{x\}$ is an open or closed set in X. If $\{x\}$ is a closed set in X then $X - \{x\}$ is an open. Since $x \notin A$ then $A \subseteq X - \{x\}$. Since A is a $G_{\beta\omega}-$closed set and $X - \{x\}$ is an open subset of X containing A, then $Cl_{\beta\omega}(A) \subseteq X - \{x\}$. Hence $x \in X - Cl_{\beta\omega}(A)$ and this a contradiction, since $x \in Cl_{\beta\omega}(A)$. If $\{x\}$ is an open set then it is $\beta\omega-$open set. Since $x \in Cl_{\beta\omega}(A)$ then we have $\{x\} \cap A \neq \emptyset$. That is, $x \in A$ and this a contradiction. Hence A is a $\beta\omega-$closed set in X. □

THEOREM 3.9. Every $g\omega-$closed set is $G_{\beta\omega}-$closed set.

PROOF. It is clear, since $Cl_{\beta\omega}(A) \subseteq Cl_{\omega}(A)$. □

The converse of above theorem is no need to be true.

EXAMPLE 3.10. In topological space (R, τ), where $\tau = \{\emptyset, R, IR \cup \{2\}\}$ and IR is a set of irrational numbers, the set of rational numbers Q is $\beta\omega-$open set. That is, IR is $\beta\omega-$closed set and thus $Cl_{\beta\omega}(IR) = IR$. Hence IR is a $G_{\beta\omega}-$closed set. Since Q is not a $\omega-$open set, then IR is not a $\omega-$closed set, that is, $Cl_{\omega}(IR) \neq IR$. Note that $IR \subseteq IR \cup \{2\}$ and $IR \cup \{2\}$ but $Cl_{\omega}(IR)$ is not subset of $R \cup \{2\}$, note that for example, $3 \in Cl_{\omega}(IR)$ and $3 \notin IR \cup \{2\}$. That is, the set IR is not $g\omega-$closed set.

A topological space (X, τ) is called anti-locally countable space [7] if each nonempty open set in X is uncountable set.

LEMMA 3.11. [7] Let (X, τ) be anti-locally countable space. Then

(1) $Int(A) = Int_{\omega}(A)$ for every $\omega-$closed set A in X.
(2) $Cl(A) = Cl_{\omega}(A)$ for every $\omega-$open set A in X.

LEMMA 3.12. For a topological space (X, τ) and $A \subseteq X$, the following hold:

(1) $Int_{\beta\omega}(X - A) = X - Cl_{\beta\omega}(A)$.
(2) $Cl_{\beta\omega}(X - A) = X - Int_{\beta\omega}(A)$.

PROOF. 1. Since $A \subseteq Cl_{\beta\omega}(A)$, then $X - Cl_{\beta\omega}(A) \subseteq X - A$. Since $Cl_{\beta\omega}(A)$ is a $\beta\omega-$closed set then $X - Cl_{\beta\omega}(A)$ is a $\beta\omega-$open set. Then

$$X - Cl_{\beta\omega}(A) = Int_{\beta\omega}[X - Cl_{\beta\omega}(A)] \subseteq Int_{\beta\omega}(X - A).$$

For the other side, let $x \in Int_{\beta\omega}(X - A)$. Then there is $\beta\omega-$open set U such that $x \in U \subseteq X - A$. Then $X - U$ is a $\beta\omega-$closed set containing A and $x \notin X - U$. Hence $x \notin Cl_{\beta\omega}(A)$, that is, $x \in X - Cl_{\beta\omega}(A)$.
2. Similar for the part(1). □

DEFINITION 3.13. A subset A of a topological space (X, τ) is called $S_{\omega}-$open set if $A \subseteq Int_{\omega}(Cl_{\omega}(A))$. The complement of $S_{\omega}-$open set is called $S_{\omega}-$closed set. The set of all $S_{\omega}-$closed sets in X denoted by $S_{\omega}C(X, \tau)$ and the set of all $S_{\omega}-$open sets in X denoted by $S_{\omega}O(X, \tau)$.

THEOREM 3.14. Let (X, τ) be anti-locally countable space and $\beta\omega O(X, \tau) = S_{\omega}O(X, \tau)$. Then

(1) $Cl(A) = Cl_{\omega}(A) = Cl_{\beta\omega}(A)$ for every $\omega-$open set A in X.

(2) $Int(A) = Int_\omega(A) = Int_{\beta\omega}(A)$ for every $\omega-$closed set A in X.

PROOF. (1) Let A be a $\omega-$open set in X. It is clear from Lemma (3.11) that $Cl(A) = Cl_\omega(A)$ and it is clear that that $Cl_{\beta\omega}(A) \subseteq Cl_\omega(A)$. Now we need to prove that $Cl_\omega(A) \subseteq Cl_{\beta\omega}(A)$. Let $x \notin Cl_{\beta\omega}(A)$. Then there is a $\beta\omega-$open set O in X such that $O \cap A = \emptyset$. Since $\beta\omega O(X,\tau) = S_\omega O(X,\tau)$, then $O \subseteq Int_\omega(Cl_\omega(O)$. Hence $Int_\omega(Cl_\omega(O)$ is a $\omega-$open set containing x and

$$\begin{aligned} Int_\omega(Cl_\omega(O)) \cap A &= Int_\omega(Cl_\omega(O)) \cap Int_\omega(A) \\ &= Int_\omega[Cl_\omega(O) \cap A] \subseteq Cl_\omega(O) \cap A \\ &\subseteq Cl_\omega(O \cap A) = Cl_\omega(\emptyset) = \emptyset. \end{aligned}$$

That is, $x \notin Cl_\omega(A)$. Hence $Cl_{\beta\omega}(A) \subseteq Cl_\omega(A)$.
(2) Similar for the part(1), by Lemma (3.12) and Lemma (3.11). □

THEOREM 3.15. Let (X,τ) be anti-locally countable space and $\beta\omega O(X,\tau) = S_\omega O(X,\tau)$. Then X is T_1- space if and only if every $G_{\beta\omega}-$closed set is a $\beta\omega-$closed set in X.

PROOF. *Necessity:* By Theorem (2.7), X is a $T_{1/2}-$ space. Then, by Theorem (3.8), every $G_{\beta\omega}-$closed set is a $\beta\omega-$closed set in X.

Sufficiency: Let $x \in X$ be an arbitrary point in X. By using Theorem (2.7), to prove that X is a T_1- space, we will prove that $\{x\}$ is a closed set in X. Suppose that $\{x\}$ is not closed set in X. Then $A = X - \{x\}$ is not open set. Then X is the only open set containing A and hence $Cl_{\beta\omega}(A) \subseteq X$, that is, A is a $G_{\beta\omega}-$closed set in X. Then, by assumption, A is a $\beta\omega-$closed set. That is, $Cl_{\beta\omega}(A) = A$. Since $X - \{x\}$ is a $\omega-$open set, then by Theorem (3.14)

$$Cl(A) = Cl_\omega(A) = Cl_{\beta\omega}(A) = A.$$

That is, $\{x\}$ is an open set and this contradicts the fact (X,τ) be anti-locally countable space. Then X is T_1-space. □

THEOREM 3.16. If A is a $G_{\beta\omega}-$closed set in a topological space (X,τ) and B is a closed set in X then $A\cap B$ is a $G_{\beta\omega}-$closed set.

PROOF. Let U be an open subset of X such that $A \cap B \subseteq U$. Since B is a closed set in X then $U \cup (X - B)$ is an open set in X. Since A is a $G_{\beta\omega}-$closed set in X and $A \subseteq U \cup (X - B)$ then $Cl_{\beta\omega}(A) \subseteq U \cup (X - B)$. Hence

$$\begin{aligned} Cl_{\beta\omega}(A \cap B) &\subseteq Cl_{\beta\omega}(A) \cap Cl_{\beta\omega}(B) \subseteq Cl_{\beta\omega}(A) \cap Cl(B) \\ &= Cl_{\beta\omega}(A) \cap B \subseteq [U \cup (X - B)] \cap B \\ &\subseteq U \cap B \subseteq U. \end{aligned}$$

Thus, $A \cap B$ is a $G_{\beta\omega}-$closed set. □

THEOREM 3.17. A subset A of a topological space (X,τ) is a $G_{\beta\omega}-$open if and only if $F \subseteq Int_{\beta\omega}(A)$ whenever $F \subseteq A$ and F is closed subset of (X,τ).

PROOF. Let A be a $G_{\beta\omega}-$open subset of X and F be a closed subset of X such that $F \subseteq A$. Then $X - A$ is a $G_{\beta\omega}-$closed set in X, $X - A \subseteq X - F$ and $X - F$ is an open subset of X. Hence Lemma (3.12), $X - Int_{\beta\omega}(A) = Cl_{\beta\omega}(X - A) \subseteq X - F$, that is, $F \subseteq Int_{\beta\omega}(A)$.
Conversely, suppose that $F \subseteq Int_{\beta\omega}(A)$ where F is a closed subset of X such that $F \subseteq A$. Then for any open subset U of X such that $X - A \subseteq U$, we have $X - U \subseteq A$ and $X - U \subseteq Int_{\beta\omega}(A)$. Then by Lemma(3.12), $X - Int_{\beta\omega}(A) = Cl_{\beta\omega}(X - A) \subseteq U$. Hence $X - A$ is a $G_{\beta\omega}-$closed (i.e., A is a $G_{\beta\omega}-$open set). □

THEOREM 3.18. If A is a $G_{\beta\omega}-$closed subset of a topological space (X,τ) then $Cl_{\beta\omega}(A) - A$ contains no nonempty closed set.

PROOF. Suppose that $Cl_{\beta\omega}(A) - A$ contains nonempty closed set F. Then

$$F \subseteq Cl_{\beta\omega}(A) - A \subseteq Cl_{\beta\omega}(A).$$

Since $A \subseteq Cl_{\beta\omega}(A)$ then $F \subseteq X - A$ and so $A \subseteq X - F$. Since A is a $G_{\beta\omega}-$closed set and $X - F$ is an open subset of X, then $Cl_{\beta\omega}(A) \subseteq X - F$ and so $F \subseteq X - Cl_{\beta\omega}(A)$. Therefore

$$F \subseteq Cl_{\beta\omega}(A) \cap (X - Cl_{\beta\omega}(A)) = \emptyset$$

and so $F = \emptyset$. Hence $Cl_{\beta\omega}(A) - A$ contains no nonempty closed set. □

COROLLARY 3.19. If A is a $G_{\beta\omega}-$closed subset of a topological space (X,τ) then $Cl_{\beta\omega}(A) - A$ is a $G_{\beta\omega}-$open set.

PROOF. By Theorem (3.18), $Cl_{\beta\omega}(A) - A$ contains no nonempty closed set and it is clear that $\emptyset \subseteq Int_{\beta\omega}(Cl_{\beta\omega}(A) - A)$ then by Theorem (3.17), $Cl_{\beta\omega}(A) - A$ is a $G_{\beta\omega}-$open set. □

THEOREM 3.20. If A is a $G_{\beta\omega}-$closed subset of a topological space (X,τ) and $B \subseteq X$. If $A \subseteq B \subseteq Cl_{\beta\omega}(A)$ then B is a $G_{\beta\omega}-$closed set.

PROOF. Let U be an open set in X such that $B \subseteq U$. Then $A \subseteq B \subseteq U$. Since A is a $G_{\beta\omega}-$closed set then $Cl_{\beta\omega}(A) \subseteq U$. Since $B \subseteq Cl_{\beta\omega}(A)$ then

$$Cl_{\beta\omega}(B) \subseteq Cl_{\beta\omega}[Cl_{\beta\omega}(A)] = Cl_{\beta\omega}(A) \subseteq U.$$

That is, B is a $G_{\beta\omega}-$closed set. □

THEOREM 3.21. Let A be a $G_{\beta\omega}-$closed subset of a topological space (X,τ). Then $A = Cl_{\beta\omega}(Int_{\beta\omega}(A))$ if and only if $Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A$ is a closed set.

PROOF. Let $Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A$ be a closed set. Since $Int_{\beta\omega}(A) \subseteq A$ and $A \subseteq Cl_{\beta\omega}(A)$, then $Cl_{\beta\omega}(Int_{\beta\omega}(A)) \subseteq Cl_{\beta\omega}(A)$. Then $Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A \subseteq Cl_{\beta\omega}(A) - A$, this implies

$$Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A \subseteq X - A.$$

Hence $A \subseteq X - (Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A)$. Since A is a $G_{\beta\omega}-$closed set and $X - (Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A)$ is an open set containing A, then $Cl_{\beta\omega}(A) \subseteq X - (Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A)$, this implies

$$Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A \subseteq X - Cl_{\beta\omega}(A).$$

Therefore

$$Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A \subseteq Cl_{\beta\omega}(A) \cap (X - Cl_{\beta\omega}(A)) = \emptyset.$$

Hence $Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A = \emptyset$, that is, $Cl_{\beta\omega}(Int_{\beta\omega}(A)) = A$. Conversely, if $A = Cl_{\beta\omega}(Int_{\beta\omega}(A))$ then $Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A = \emptyset$ and hence $Cl_{\beta\omega}(Int_{\beta\omega}(A)) - A$ is a closed set. □

THEOREM 3.22. Let Y be an open subset of a topological space (X,τ). If A is a $\beta\omega-$open set in (X,τ) then $A \cap Y$ is a $\beta\omega-$open set in $(Y, \tau|_Y)$.

PROOF. Since A be a $\beta\omega-$open set in (X,τ), then $A \subseteq Cl(Int_\omega(Cl(A)))$. Since Y is an open set, then by Theorem (2.1),

$$\begin{aligned} A\cap Y &= (A\cap Y)\cap Y \subseteq [(Cl(Int_\omega(Cl(A))))\cap Y]\cap Y \\ &\subseteq Cl[Int_\omega(Cl(A))\cap Y]\cap Y \\ &= Cl|_Y[Int_\omega(Cl(A))\cap Y] \\ &= Cl|_Y[Int_\omega(Cl(A))\cap Int_\omega(Y)] \\ &= Cl|_Y[Int_\omega(Cl(A)\cap Y)] \\ &= Cl|_Y[Int_\omega(Cl(A)\cap Y\cap Y)] \\ &\subseteq Cl|_Y[Int_\omega(Cl(A\cap Y)\cap Y)] \\ &= Cl|_Y[Int_\omega(Cl|_Y(A\cap Y))] \\ &\subseteq Cl|_Y[Int_\omega|_Y(Cl|_Y(A\cap Y))]. \end{aligned}$$

Therefore $A\cap Y$ is a $\beta\omega-$open set in $(Y,\tau|_Y)$. □

THEOREM 3.23. Let Y be an open subset of a topological space (X,τ). If A is a $\beta\omega-$open set in $(Y,\tau|_Y)$ then A is a $\beta\omega-$open set in (X,τ).

PROOF. Since A is a $\beta\omega-$open set in $(Y,\tau|_Y)$ and since Y is an open set, then

$$\begin{aligned} A &\subseteq Cl|_Y(Int_\omega|_Y(Cl|_Y(A))) = Cl|(Int_\omega|_Y(Cl|_Y(A)))\cap Y \\ &\subseteq Cl|(Int_\omega|_Y(Cl|_Y(A))\cap Y) = Cl|(Int_\omega(Cl|_Y(A))\cap Y) \\ &= Cl|(Int_\omega(Cl|_Y(A)\cap Y)) = Cl|(Int_\omega(Cl|_Y(A))) \\ &= Cl|(Int_\omega(Cl(A)\cap Y)) \subseteq Cl|(Int_\omega(Cl(A\cap Y))) \\ &= Cl|(Int_\omega(Cl(A))). \end{aligned}$$

Therefore A is a $\beta\omega-$open set in X. □

THEOREM 3.24. Let Y be an open subset of a topological space (X,τ) and A be a subset of Y. Then $Cl_{\beta\omega}|_Y(A) = Cl_{\beta\omega}(A)\cap Y$.

PROOF. Let $x\in Cl_{\beta\omega}|_Y(A)$ and G be a $\beta\omega-$open set in X containing x. By Theorem (3.22), $G\cap Y$ is a $\beta\omega-$open set in Y containing x and since $x\in Cl_{\beta\omega}|_Y(A)$, then $G\cap A = (G\cap Y)\cap A\neq\emptyset$. Then $x\in Cl_{\beta\omega}(A)$ and since $x\in Y$, this implies $x\in Cl_{\beta\omega}(A)\cap Y$. That is, $Cl_{\beta\omega}|_Y(A)\subseteq Cl_{\beta\omega}(A)\cap Y$. On the other side, let $x\in Cl_{\beta\omega}(A)\cap Y$ and O be a $\beta\omega-$open set in Y containing x. By Theorem (3.23), $O = G\cap Y$ for some $\beta\omega-$open set G in X. Since $x\in Cl_{\beta\omega}(A)$, then $G\cap A\neq\emptyset$ and so $(G\cap Y)\cap A\neq\emptyset$, since $x\in Y$. Hence $O\cap A\neq\emptyset$, that is, $x\in Cl_{\beta\omega}|_Y(A)$. Hence $Cl_{\beta\omega}(A)\cap Y\subseteq Cl_{\beta\omega}|_Y(A)$. □

THEOREM 3.25. Let Y be an open subspace of a topological space (X,τ) and $A\subseteq Y$. If A is a $G_{\beta\omega}-$closed subset in X then A is a $G_{\beta\omega}-$closed set in Y.

PROOF. Let O be an open subset in Y such that $A\subseteq O$. Then $O = U\cap Y$ for some open set U in X and so $A\subseteq U$. Since A is a $G_{\beta\omega}-$closed subset of X, then $Cl_{\beta\omega}(A)\subseteq U$. By Theorem (3.24), $Cl_{\beta\omega}|_Y(A) = Cl_{\beta\omega}(A)\cap Y\subseteq U\cap Y = O$. Hence A is a $G_{\beta\omega}-$closed set in Y. □

THEOREM 3.26. Let Y be an open subspace of a topological space (X,τ) and $A\subseteq Y$. If A is a $G_{\beta\omega}-$closed subset in Y and Y is $\beta\omega-$closed in X then A is a $G_{\beta\omega}-$closed set in X.

PROOF. Let U be an open subset in X such that $A\subseteq U$. Then $A\subseteq U\cap Y$ and $U\cap Y$ is open set in Y. Since A is a $G_{\beta\omega}-$closed subset in Y, then $Cl_{\beta\omega}|_Y(A)\subseteq U\cap Y$. Since Y is an open set in X and it is $\beta\omega-$closed in X then By Theorem (3.24),

$$\begin{aligned} Cl_{\beta\omega}(A) &= Cl_{\beta\omega}(A\cap Y)\subseteq Cl_{\beta\omega}(A)\cap Cl_{\beta\omega}(Y) \\ &= Cl_{\beta\omega}(A)\cap Y \\ &= Cl_{\beta\omega}|_Y(A)\subseteq U\cap Y\subseteq U. \end{aligned}$$

Hence A is a $G_{\beta\omega}-$closed set in X. □

4. $\beta\omega-$CONTINUOUS FUNCTIONS

DEFINITION 4.1. A function $f:(X,\tau)\to(Y,\rho)$ of a topological space (X,τ) into a space (Y,ρ) is called $\beta\omega-$*continuous* if $f^{-1}(U)$ is a $\beta\omega-$open set in X for every open set U in Y.

THEOREM 4.2. A function $f:(X,\tau)\to(Y,\rho)$ of a topological space (X,τ) into a space (Y,ρ) is $\beta\omega-$continuous if and only if $f^{-1}(F)$ is a $\beta\omega-$closed set in X for every closed set F in Y.

THEOREM 4.3. Every $\omega-$continuous function is $\beta\omega-$continuous function.

The converse of the last theorem is no need to be true.

EXAMPLE 4.4. Let $f:(R,\tau)\to(R,\rho)$ be a function defined by $f(r)=r$, where

$$\tau=\{\emptyset,R\} \text{ and } \rho=\{\emptyset,R,\{2\}\}.$$

The function f is a $\beta\omega-$continuous, since $f^{-1}(\{2\})=\{2\}$ and $f^{-1}(R)=R$ are $\beta\omega-$open sets in (R,τ). The function f is not $\omega-$continuous, since $f^{-1}(\{2\})=\{2\}$ is not $\omega-$open set in (R,τ).

THEOREM 4.5. If $f:(X,\tau)\to(Y,\rho)$ is a $\beta\omega-$continuous function then for each $x\in X$ and each open set U in Y with $f(x)\in U$, there exists a $\beta\omega-$open set V in X such that $x\in V$ and $f(V)\subseteq U$.

PROOF. Let $x\in X$ and U be any open set in Y containing $f(x)$. Put $V=f^{-1}(U)$. Since f is a $\beta\omega-$continuous then V is a $\beta\omega-$open set in X such that $x\in V$ and $f(V)\subseteq U$.
conversely, Let U be any open set in Y. Let $x\in f^{-1}(U)$. Then $f(x)\in U$ and hence by the hypothesis, there exists a $\beta\omega-$open set V in X such that $x\in V$ and $f(V)\subseteq U$. Hence $x\in V\subseteq f^{-1}(U)$, that is, $f^{-1}(U)$ is a $\beta\omega-$open set in X. That is, f is a $\beta\omega-$continuous. □

THEOREM 4.6. Let $f:(X,\tau)\to(Y,\rho)$ be a function of a space (X,τ) into a space (Y,ρ). Then f is a $\beta\omega-$continuous if and only if $f[Cl_{\beta\omega}(A)]\subseteq Cl(f(A))$ for all $A\subseteq X$.

PROOF. Let f be a $\beta\omega-$continuous and A be any subset of X. Then $Cl(f(A))$ is a closed set in Y. Since f is a $\beta\omega-$continuous then by Theorem (4.2), $f^{-1}[Cl(f(A))]$ is a $\beta\omega-$closed set in X. That is,

$$Cl_{\beta\omega}\left[f^{-1}[Cl(f(A))]\right]=f^{-1}[Cl(f(A))].$$

Since $f(A)\subseteq Cl(f(A))$ then $A\subseteq f^{-1}[Cl(f(A))]$. This implies,

$$Cl_{\beta\omega}(A)\subseteq Cl_{\beta\omega}\left[f^{-1}[Cl(f(A))]\right]=f^{-1}[Cl(f(A))].$$

Hence $f[Cl_{\beta\omega}(A)]\subseteq Cl(f(A))$.
Conversely, let H be any closed set in Y, that is, $Cl(H)=H$. Since $f^{-1}(H)\subseteq X$. Then by the hypothesis,

$$f\left[Cl_{\beta\omega}[f^{-1}(H)]\right]\subseteq Cl[f(f^{-1}(H))]\subseteq Cl(H)=H.$$

This implies, $Cl_{\beta\omega}[f^{-1}(H)]\subseteq f^{-1}(H)$. Hence $Cl_{\beta\omega}[f^{-1}(H)]=f^{-1}(H)$, that is, $f^{-1}(H)$ is a $\beta\omega-$closed set in X. Therefore f is a $\beta\omega-$continuous. □

THEOREM 4.7. Let $f:(X,\tau)\rightarrow(Y,\rho)$ be a function of a space (X,τ) into a space (Y,ρ). Then f is $\beta\omega-$continuous if and only if $Cl_{\beta\omega}(f^{-1}(B))\subseteq f^{-1}(Cl(B))$ for all $B\subseteq Y$.

PROOF. Let f be a $\beta\omega-$continuous and B be any subset of Y. Then $Cl(B)$ is a closed set in Y. Since f is a $\omega-$continuous then by Theorem(4.2), $f^{-1}[Cl(B)]$ is a $\beta\omega-$closed set in X. That is, $Cl_{\beta\omega}\left[f^{-1}[Cl(B)]\right]=f^{-1}[Cl(B)]$. Since $B\subseteq Cl(B)$ then $f^{-1}(B)\subseteq f^{-1}[Cl(B)]$. This implies,

$$Cl_{\beta\omega}(f^{-1}(B))\subseteq Cl_{\beta\omega}\left[f^{-1}[Cl(B)]\right]=f^{-1}[Cl(B)].$$

Hence $Cl_{\beta\omega}(f^{-1}(B))\subseteq f^{-1}[Cl(B)]$.
Conversely, Let H be any closed set in Y, that is, $Cl(H)=H$. Since $H\subseteq Y$. Then by the hypothesis,

$$Cl_{\beta\omega}(f^{-1}(H))\subseteq f^{-1}(Cl(H))=f^{-1}(H).$$

This implies, $Cl_{\beta\omega}[f^{-1}(H)]\subseteq f^{-1}(H)$. Hence $Cl_{\beta\omega}[f^{-1}(H)]=f^{-1}(H)$, that is, $f^{-1}(H)$ is a $\beta\omega-$closed set in X. Hence f is a $\beta\omega-$continuous. □

THEOREM 4.8. Let $f:(X,\tau)\rightarrow(Y,\rho)$ be a function of a space (X,τ) into a space (Y,ρ). Then f is $\beta\omega-$continuous if and if $f^{-1}(Int(B))\subseteq Int_{\beta\omega}[f^{-1}(B)]$ for all $B\subseteq Y$.

PROOF. Let f be a $\beta\omega-$continuous and B be any subset of Y. Then $Int(B)$ is an open set in Y. Since f is a $\omega-$continuous then $f^{-1}[Int(B)]$ is a $\beta\omega-$open set in X. That is, $Int_{\beta\omega}\left[f^{-1}[Int(B)]\right]=f^{-1}[Int(B)]$. Since $Int(B)\subseteq B$ then $f^{-1}[Int(B)]\subseteq f^{-1}(B)$. This implies,

$$f^{-1}[Int(B)]=Int_{\beta\omega}\left[f^{-1}[Int(B)]\right]\subseteq Int_{\beta\omega}(f^{-1}(B)).$$

Hence $f^{-1}(Int(B))\subseteq Int_{\beta\omega}[f^{-1}(B)]$.
Conversely, let U be any open set in Y, that is, $Int(U)=U$. Since $U\subseteq Y$. Then by the hypothesis,

$$f^{-1}(U)=f^{-1}(Int(U))\subseteq Int_{\beta\omega}[f^{-1}(U)].$$

This implies, $f^{-1}(U)\subseteq Int_{\beta\omega}[f^{-1}(U)]$. Hence $f^{-1}(U)=Int_{\beta\omega}[f^{-1}(U)]$, that is, $f^{-1}(U)$ is a $\beta\omega-$open set in X. Hence f is $\beta\omega-$continuous. □

DEFINITION 4.9. A function $f:(X,\tau)\rightarrow(Y,\rho)$ of a topological space (X,τ) into a space (Y,ρ) is called *generalized $\beta\omega-$continuous* (simply $G_{\beta\omega}-$continuous) *function*, if $f^{-1}(U)$ is a $G_{\beta\omega}-$open set in X for every open set U in Y.

THEOREM 4.10. A function $f:(X,\tau)\rightarrow(Y,\rho)$ of a topological space (X,τ) into a space (Y,ρ) is $G_{\beta\omega}-$continuous if and only if $f^{-1}(F)$ is a $G_{\beta\omega}-$closed set in X for every closed set F in Y.

THEOREM 4.11. Every $\beta\omega-$continuous function is $G_{\beta\omega}-$continuous function.

The converse of the last theorem is no need to be true.

EXAMPLE 4.12. Let $f:(R,\tau)\rightarrow(R,\rho)$ be a function defined by $f(r)=r$, where

$$\tau=\{\emptyset,R,R-\{2,3\}\}\text{ and }\rho=\{\emptyset,R,\{2\}\}.$$

The function f is a $G_{\beta\omega}-$continuous, since $f^{-1}(\{2\})=\{2\}$ and $f^{-1}(R)=R$ are $G_{\beta\omega}-$open sets in (R,τ). The function f is not $\beta\omega-$continuous, since $f^{-1}(\{2\})=\{2\}$ is not $\beta\omega-$open set in (R,τ).

THEOREM 4.13. Let $f:(X,\tau)\rightarrow(Y,\rho)$ be a function of a $T_{1/2}-$space (X,τ) into a space (Y,ρ). If f is a $G_{\beta\omega}-$continuous then it is a $\beta\omega-$continuous.

PROOF. Let $f:(X,\tau)\rightarrow(Y,\rho)$ be a $G_{\beta\omega}-$continuous function and U be any open set in Y. Then $f^{-1}(U)$ is a $G_{\beta\omega}-$open set in X. Since X is a $T_{1/2}-$space then by Theorem (3.8), $f^{-1}(U)$ is a $\beta\omega-$open set in X. That is, f is a $\beta\omega-$continuous function. □

THEOREM 4.14. Every $g\omega-$continuous function is $G_{\beta\omega}-$continuous function.

PROOF. Let $f:(X,\tau)\rightarrow(Y,\rho)$ be a $g\omega-$continuous function and U be any open set in Y. Then $f^{-1}(U)$ is a $g\omega-$open set in X and by Theorem (3.9), $f^{-1}(U)$ is a $G_{\beta\omega}-$open set in X. That is, f is a $G_{\beta\omega}-$continuous function. □

The converse of the last theorem is no need to be true.

EXAMPLE 4.15. Let $f:(R,\tau)\rightarrow(R,\rho)$ be a function defined by

$$f(x)\;=\;\begin{cases}2, & x\in IR\\ x, & x\notin IR\end{cases}$$

where

$$\tau=\{\emptyset,R,IR\cup\{2\}\}\text{ and }\rho=\{\emptyset,R,\{2\}\},$$

IR is a set of irrational numbers. The function f is a $G_{\beta\omega}-$continuous, since $f^{-1}(\{2\})=IR$ and $f^{-1}(R)=R$ are $G_{\beta\omega}-$open sets in (R,τ). The function f is not $g\omega-$continuous, since $f^{-1}(\{2\})=IR$ is not $g\omega-$open set in (R,τ).

THEOREM 4.16. If $f:(X,\tau)\rightarrow(Y,\rho)$ is a $G_{\beta\omega}-$continuous function then for each $x\in X$ and each open set U in Y with $f(x)\in U$, there exists a $G_{\beta\omega}-$open set V in X such that $x\in V$ and $f(V)\subseteq U$.

PROOF. Let $x\in X$ and U be any open set in Y containing $f(x)$. Put $V=f^{-1}(U)$. Since f is a $G_{\beta\omega}-$continuous then V is a $G_{\beta\omega}-$open set in X such that $x\in V$ and $f(V)\subseteq U$. □

The converse of the last theorem need not be true.

EXAMPLE 4.17. Let $f:(R,\tau)\rightarrow(R,\rho)$ be a function defined by

$$f(x)\;=\;\begin{cases}2, & x\in\{2,3\}\\ x, & x\notin\{2,3\}\end{cases}$$

where

$$\tau=\{\emptyset,R,R-\{2,3\}\}\text{ and }\rho=\{\emptyset,R,\{2\}\}.$$

The function f is not $G_{\beta\omega}-$continuous, since $f^{-1}(\{2\})=\{2,3\}$ is not $G_{\beta\omega}-$open set in (R,τ). On the other hand, for all $x\in R$, $\{x\}$ is a $G_{\beta\omega}-$open set in (R,τ).

5. REFERENCES

[1] F. Helen, 1968, Introduction to General Topology, Boston: University of Massachusetts.

[2] H. Z. Hdeib, $w-$closed mappings, Revista Colombiana de Matematicas, 16 (1982), 65-78.

[3] J. Dontchev and H. Maki, On $\theta-$generalized closed sets, Int. J. Math. Math. Sci., 22 (1999), 239-249.

,

[4] K. Al-Zoubi, On generalized $\omega-$closed sets, International Journal of Mathematics and Mathematical Sciences, 13 (2005), 20112021.

[5] M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, $\beta-$open sets and $\beta-$continuous mapping, Bull. Fac. Sci. Assiut Univ., 12 (1983), 77-90.

[6] N. Levine, Generalized closed sets in topology, Rend. Cric. Mat.Palermo, 2 (1970), 89-96.

[7] T. Noiri, A. Al-omari and M. Noorani, Weak forms of $\omega-$open sets and decompositions of continuity, European Journal of Pure and Applied Mathematics 1, (2009), 73-84.